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GRADIENT-VELOCITY FRAMEWORK FOR APERIODIC ROBUST STABILITY OF SATELLITE COMMUNICATION TRACKING LOOPS UNDER PARAMETRIC UNCERTAINTY

ПАРАМЕТРЛІК АНЫҚТАЛМАҒАНДЫҚ ЖАҒДАЙЫНДА СПУТНИКТИК БАЙЛАНЫС БАҚЫЛАУ КОНТУРЛАРЫНЫҢ АПЕРИОДТЫ РОБАСТЫ ТҰРАҚТЫЛЫҒЫНА АРНАЛҒАН ГРАДИЕНТТИ-ЖЫЛДАМДЫҚ ТӘСІЛ

ГРАДИЕНТНО-СКОРОСТНОЙ ПОДХОД К АПЕРИОДИЧЕСКОЙ РОБАСТНОЙ УСТОЙЧИВОСТИ КОНТУРОВ СЛЕЖЕНИЯ СПУТНИКОВОЙ СВЯЗИ ПРИ ПАРАМЕТРИЧЕСКОЙ НЕОПРЕДЕЛЁННОСТИ

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ABSTRACT

This study develops a gradient-velocity framework for evaluating aperiodic robust stability in normalized satellite-communication tracking loops with bounded parametric uncertainty. The model represents amplitude-regulation, timing-synchronization, and carrier-tracking errors in state space. A common quadratic Lyapunov certificate is checked at the endpoints of the uncertainty interval, while the gradient-velocity condition is used to verify sign-preserving, non-oscillatory convergence. Two- and three-state numerical examples are examined for μ from 0.2 to 0.8. For the nominal two-state case, the settling time is 3.07 s, the integral absolute error is 0.784, and no undershoot occurs. The third-order extension also remains stable and monotonic throughout the admissible interval. The scientific novelty lies in separating the robust-stability certificate from the explicit aperiodicity verification of the state trajectories. The framework provides a reproducible preliminary-design tool for communication tracking loops and clarifies the relationship between robust decay and transient shape. Validation on nonlinear, delayed, coupled, and measured communication data remains necessary.



Түйінді сөздер:

робасты тұрақтылық, спутниктік байланыс жүйелері, Ляпунов функциясы, параметрлік анықталмағандық, апериодтық өтпелі процесс, градиентті-жылдамдық әдісі, күй кеңістігі моделі.

ТҮЙІНДЕМЕ

Жұмыста шектеулі параметрлік анықталмағандық жағдайындағы спутниктік байланыс бақылау контурларының апериодты робасты тұрақтылығын бағалауға арналған градиентті-жылдамдық тәсіл әзірленді. Күй кеңістігінде амплитуданы реттеу, уақыттық синхрондау және тасымалдаушы жиілікті бақылау қателері модельденеді. Ортақ квадраттық Ляпунов функциясы анықталмағандық интервалының шекараларында тексеріліп, градиентті-жылдамдық шарты таңбасы өзгермейтін және тербеліссіз жинақталуды растау үшін қолданылады. μ мәні 0,2-ден 0,8-ге дейін болғанда екінші және үшінші ретті модельдер зерттелді. Екінші ретті номинал модель үшін орнығу уақыты 3,07 с, интегралдық абсолют қате 0,784 болды, ал асып кету байқалмады. Үшінші ретті модель де бүкіл рұқсат етілген аралықта тұрақты және монотонды күйін сақтады. Салыстырмалы талдау тәсілдің тұрақтылықпен бірге апериодтылық талап етілгенде тиімді екенін көрсетті. Болашақта әдісті кідірісі бар бейсызық модельдерде және нақты байланыс деректерінде тексеру қажет.

Ключевые слова:

робастная устойчивость, спутниковые системы связи, функция Ляпунова, параметрическая неопределённость, апериодический переходный процесс, градиентно-скоростной метод, модель в пространстве состояний.

АННОТАЦИЯ

В работе разработан градиентно-скоростной подход к оценке апериодической робастной устойчивости нормированных контуров слежения спутниковой связи при ограниченной параметрической неопределённости. В пространстве состояний моделируются ошибки регулирования амплитуды, временной синхронизации и слежения за несущей. Общая квадратичная функция Ляпунова проверяется на границах интервала неопределённости, а градиентно-скоростное условие используется для подтверждения сходимости без смены знака и колебаний. Исследованы модели второго и третьего порядка при значениях μ от 0,2 до 0,8. Для номинальной модели второго порядка время установления составило 3,07 с, интегральная абсолютная ошибка – 0,784, перерегулирование отсутствует. Модель третьего порядка также сохраняет устойчивость и монотонность во всём допустимом интервале. Сравнение с эталонными реализациями показывает практическую ценность подхода при одновременном требовании устойчивости и апериодичности. Необходима дальнейшая проверка на нелинейных моделях с задержками и реальными данными связи.

INTRODUCTION

Satellite communication has become a core element of broadband access, navigation, remote sensing, and geographically distributed digital services. New Space architectures increase the number of low-Earth-orbit links and therefore the number of acquisition, synchronization, and tracking operations that must remain reliable during rapid channel changes (Kodheli et al., 2021). Recent work on timing synchronization, robust precoding, and risk-aware beamforming confirms that Doppler shifts, imperfect channel-state information, and gain uncertainty directly affect link performance (Kim & Park, 2023; Liu et al., 2022; Alsenwi et al., 2024).

At the control-loop level, these effects appear as variations in effective gain, damping, and coupling between amplitude, timing, and carrier-tracking errors. A loop can be asymptotically stable and still exhibit sign reversals or oscillatory transients. Such behavior increases acquisition time and may produce temporary degradation of signal quality. Robust stability analysis should therefore distinguish between convergence to equilibrium and the quality of the path followed during that convergence.

Quadratic Lyapunov functions provide a transparent certificate for uncertain linear systems, and linear matrix inequality (LMI) methods make many robust-control conditions computationally tractable. Yet the inequality $dV/dt < 0$ alone does not imply that every communication-error state is monotonic. Aperiodicity requires an additional property of the selected state-space realization, such as sign preservation and real negative modal behavior. This distinction is important because a favorable energy decay curve can coexist with undesirable oscillations in individual states.

The objective of this study is to formulate and test a gradient-velocity framework that combines a common Lyapunov certificate with an explicit aperiodicity check for normalized satellite-communication tracking loops under affine parametric uncertainty. The framework is evaluated on reproducible second- and third-order examples. The interval $[0.2, 0.8]$ for μ is treated as an admissible normalized design range rather than as a tolerance identified for a particular modem.

The contribution is threefold. First, the robust certificate is reduced to endpoint tests because the symmetric Lyapunov matrix is affine in μ . Second, the stability result is separated from the componentwise monotonicity argument, preventing the usual over interpretation of $dV/dt < 0$. Third, the numerical model, benchmark matrices, initial conditions, and performance definitions are stated explicitly so that the reported settling time, undershoot, and integral absolute error can be reproduced.

LITERATURE REVIEW

The direct Lyapunov method remains the basic analytical route for certifying stability without solving the state equations explicitly (Khalil, 2015). Matrix formulations support compact treatment of multivariable systems (Bernstein, 2009), while common quadratic certificates and LMIs provide sufficient conditions over polytopic or norm-bounded uncertainty sets (Boyd et al., 1994; Zhou et al., 1996). These tools establish stability rigorously, but transient requirements must be introduced through pole regions, decay-rate bounds, overshoot constraints, or a separate analysis of the state response.

Recent aerospace studies combine Lyapunov analysis with LMI synthesis, adaptive control, observers, and finite-time techniques. Examples include fault-tolerant spacecraft attitude control (Gao et al., 2021; Giuseppi et al., 2022), adaptive tracking under model and environmental uncertainty (Cho, 2023), observer-based parametric control of flexible spacecraft (Zhang et al., 2024), negative-imaginary output feedback for uncertain satellite dynamics (Choudhary & Chokkadi, 2024), and hardware-in-the-loop validation of a Lyapunov-LMI launch-vehicle controller (Brusnicki et al., 2026).

Communication-oriented research treats uncertainty mainly through synchronization, channel estimation, precoding, and beamforming. Robust timing acquisition under a low-Earth-orbit channel was examined by Kim and Park (2023). Liu et al. (2022) studied downlink precoding under per-antenna constraints, and Alsenwi et al. (2024) proposed risk-aware robust beamforming for massive-MIMO satellite links. These studies address communication performance directly, but they do not replace a state-space transient analysis of the underlying tracking loops.

The gradient-velocity method of Lyapunov vector functions was developed to connect the Lyapunov gradient with the velocity field and to formulate desired transient behavior directly from closed-loop matrix coefficients (Beisenbi et al., 2021). Its later application to adaptive and nonlinear spacecraft control illustrates the method's broader use under uncertainty (Beisenbi et al., 2025). The present study does not claim that this construction supersedes classical Lyapunov theory; it uses the gradient-velocity interpretation as an operational framework for jointly checking robust decay and aperiodicity.

The unresolved issue is not the existence of another asymptotic-stability inequality, but a transparent way to show when a specific uncertain tracking-loop realization is both stable and non-oscillatory. For this reason, the analysis below reports the Lyapunov margin and the state trajectories separately. Claims about satellite communication are limited to normalized small-signal tracking-loop errors; no assertion is made that the example represents a complete radio-frequency link.

MATERIALS AND METHODS

A normalized small-signal tracking subsystem is represented by a continuous-time linear state-space model with bounded affine uncertainty. In the second-order realization, $x_1(t)$ denotes the amplitude-regulation error and $x_2(t)$ denotes the symbol-timing error. The scalar μ is a normalized effective loop-gain parameter that combines residual gain and channel-dependent variation after linearization.

$$\dot{x}(t) = A(\mu)x(t) + Bu(t),$$

where $x(t)$ is a two-dimensional real state vector, $u(t)$ is a scalar real-valued external correction input, $A(\mu)$ is the uncertain closed-loop matrix, and B is the input matrix. Robust stability is examined for the autonomous error dynamics $u=0$. The reproducible matrices are defined as follows:

$$\begin{aligned} A(\mu) &= A_0 + \mu\Delta A, \quad \mu \in [0.2, 0.8], \\ A_0 &= \begin{bmatrix} -1.10 & 0 \\ 0 & -1.50 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} -0.35 & 0 \\ 0 & -0.60 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ x(0) &= [1.0 \quad 0.8]^T, \end{aligned}$$

The assumptions are: (i) μ is constant during one simulation and belongs to $[0.2, 0.8]$; (ii) the affine model is valid in a neighborhood of the locked operating point; (iii) the error states are available for analysis; (iv) $u=0$ during stability verification; and (v) the interval is a representative design domain, not a tolerance estimated from flight or modem data. Because the corrected nominal matrix is diagonal, the two error loops are decoupled in this illustrative realization.

A quadratic Lyapunov function is selected in the form

$$V(x) = x^T P x, \quad P = P^T > 0,$$

Its derivative along the system trajectories is

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T [A^T(\mu)P + PA(\mu)]x + 2x^T P B u,$$

For the autonomous stability analysis, $u=0$. The gradient-velocity formulation explicitly relates the gradient of the Lyapunov function to the state velocity:

$$\nabla V(x) = 2Px, \quad \dot{V}(x) = \nabla V(x)^T \dot{x}(t),$$

Robust asymptotic stability is guaranteed if a common matrix P exists such that

$$A^T(\mu)P + PA(\mu) \leq -\alpha I, \quad \alpha > 0, \quad \forall \mu \in [0.2, 0.8],$$

Consequently,

$$\dot{V}(x) \leq -\alpha \|x\|^2 < 0, \quad x \neq 0.$$

The strict decrease of $V(x)$ establishes robust asymptotic stability. Aperiodicity is verified separately. For the diagonal realization $A(\mu) = \text{diag}[-a_1(\mu), -a_2(\mu)]$ with $a_i(\mu) > 0$, each state is sign preserving and monotonic. Thus the Lyapunov certificate and the non-oscillatory transient statement rely on distinct, explicitly stated conditions.

NUMERICAL IMPLEMENTATION

Numerical trajectories were evaluated from $t=0$ to $t=10$ s on a 0.01 s grid using the matrix exponential and cross-checked with the MATLAB R2024a ode45 settings retained from the original manuscript (relative tolerance 10-6 and absolute tolerance 10-8). The initial condition was $x(0)=[1.0,0.8]^T$. Robustness was evaluated for $\mu=\{0.2,0.4,0.6,0.8\}$; nominal plots use $\mu=0.5$. Settling time is the first time after which $|x_1(t)|$ remains within 2% of $|x_1(0)|$. Undershoot is the largest negative excursion of x_1 relative to $x_1(0)$, and $IAE = \int_0^{10} |x_1(t)| dt$.

Two stable second-order matrices are retained as transparent transient-response benchmarks:

$$A_C = \begin{bmatrix} -0.65 & 1.55 \\ -1.55 & -0.65 \end{bmatrix}, \quad A_{LMI} = \begin{bmatrix} -0.90 & 0.65 \\ -0.65 & -0.90 \end{bmatrix}.$$

They are evaluated with the same initial condition $x(0)=[1.0,0.8]^T$, simulation horizon, time grid, and definitions of settling time, undershoot, and IAE as the gradient-velocity realization. The comparison therefore isolates differences in transient geometry; it is not presented as a head-to-head controller-synthesis test because the three matrices are prescribed closed-loop realizations rather than controllers synthesized for one common plant.

ENDPOINT ROBUSTNESS CERTIFICATE

For an affine matrix $A(\mu)$ and fixed P , the symmetric matrix $A(\mu)^T P + P A(\mu)$ is affine in μ . Negative definiteness at both vertices therefore implies negative definiteness throughout the convex interval. The endpoint condition used in this study is

$$A(\mu_k)^T P + P A(\mu_k) \leq -\alpha I, \quad \mu_k \in \{0.2, 0.8\}.$$

For the corrected two-state model, $P = I_2$. At $\mu=0.2$ the eigenvalues of A are -1.17 and -1.62, while at $\mu=0.8$ they are -1.38 and -1.98. Consequently, the symmetric matrix $A^T(\mu) + A(\mu)$ is bounded above by $-2.34I_2$ over the full interval. This value is a conservative common decay margin obtained at the slowest endpoint.

THIRD-ORDER EXTENSION

To verify that the procedure is not tied to two states, a third normalized error state is added for carrier-phase tracking. The decoupled third-order small-signal realization is

$$\begin{aligned} \dot{\mathbf{x}}_3(t) &= A_3(\mu) \mathbf{x}_3(t), \quad A_3(\mu) = A_{30} + \mu \Delta A_3. \\ A_{30} &= \begin{bmatrix} -1.00 & 0 & 0 \\ 0 & -1.25 & 0 \\ 0 & 0 & -1.55 \end{bmatrix}, \quad \Delta A_3 = \begin{bmatrix} -0.25 & 0 & 0 \\ 0 & -0.30 & 0 \\ 0 & 0 & -0.35 \end{bmatrix}, \quad \mathbf{x}_3(0) \\ &= [1.00 \quad 0.75 \quad 0.50]^T. \end{aligned}$$

All diagonal coefficients remain strictly negative for every μ in $[0.2, 0.8]$. With $P_3 = I_3$, the worst-case common-Lyapunov bound is

$$P_3 = I_3, \quad \dot{V}_3(\mathbf{x}_3) = \mathbf{x}_3^T [A_3(\mu)^T + A_3(\mu)] \mathbf{x}_3 \leq -2.10 \|\mathbf{x}_3\|_2^2.$$

Because every state has the closed-form response shown below, positive initial errors remain positive and decrease monotonically. This supplies the aperiodicity statement that is not implied by $dV/dt < 0$ alone.

$$x_i(t) = x_i(0) \exp[-a_i(\mu)t], \quad a_i(\mu) > 0.$$

RESULTS AND DISCUSSION

The first numerical experiment uses $\mu=0.5$, for which $A = \text{diag}(-1.275, -1.80)$. Both states therefore decay exponentially without a sign change. The slow state x_1 determines the 2% settling time, whereas x_2 converges more rapidly.

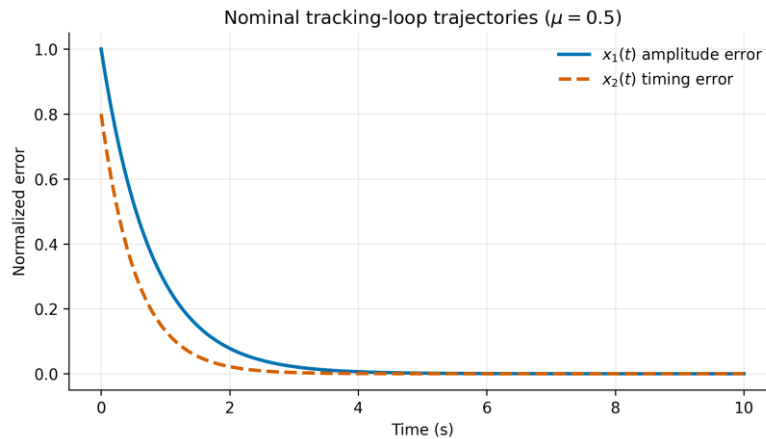


Figure 1. Nominal state trajectories of the two-state tracking-loop model

Note – compiled by the authors

Figure 1 confirms the predicted sign-preserving decay. The nominal x_1 trajectory reaches the 2% band in 3.07 s, while x_2 settles sooner because its decay rate is 1.80 s^{-1} . No state crossing or oscillatory segment is present.

Within the stated small-signal interpretation, a monotonic error response avoids repeated corrections during acquisition. The result should be read as a property of the normalized tracking-loop model, not as direct evidence of improved end-to-end bit error rate or link availability.

EVOLUTION OF THE LYAPUNOV FUNCTION

The Lyapunov function $V = x^T x$ is evaluated along the same nominal trajectory. Its monotonic decrease provides a numerical check of the analytical matrix inequality and exposes any implementation error in the state equations.

At $\mu = 0.5$, $dV/dt = -2.55x_1^2 - 3.60x_2^2 < 0$ for every nonzero state. Figure 2 is consistent with this expression: V decreases from 1.64 to a value below 10^{-9} over the simulated interval.

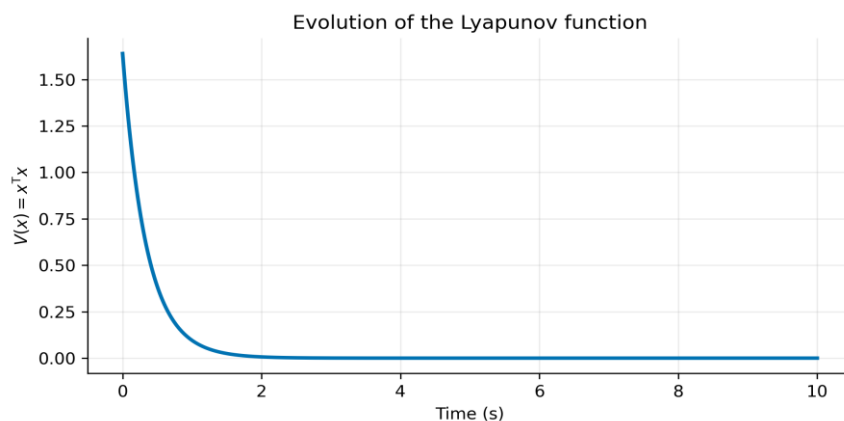


Figure 2. Evolution of the Lyapunov function over time

Note – compiled by the authors

The curve verifies robust energy decay for the nominal case. Aperiodicity, however, follows from the diagonal negative state matrix and the sign-preserving exponential solution, not from the Lyapunov curve alone.

COMPARATIVE ANALYSIS WITH BENCHMARK REALIZATIONS

The benchmark matrices were selected not as optimal controllers, but as stable second-order closed-loop realizations with deliberately different transient geometries. The classical benchmark has eigenvalues $-0.65 \pm 1.55i$ and therefore produces a pronounced oscillatory response, whereas the LMI-type benchmark has eigenvalues $-0.90 \pm 0.65i$ and represents a more strongly damped oscillatory response. At $\mu=0.5$, the gradient-velocity realization has the real negative eigenvalues -1.275 and -1.80 . This selection makes it possible to isolate the aperiodicity property under matched test conditions.

Table 1 reports the metrics computed from $x_1(t)$ using the same initial condition, common 0-10 s interval, time grid, and performance definitions. The values correspond exactly to the matrices given in the Methods section. Accordingly, the comparison is restricted to transient-response characteristics and is not interpreted as a universal comparison of controller-synthesis methods.

Table 1. Quantitative comparison of transient-response indicators

Indicator	Classical Lyapunov	LMI-based	Gradient-velocity
Settling time, s	6.35	3.07	3.07
Undershoot, %	28.2	1.4	0.0
IAE (0–10 s)	1.415	1.213	0.784
Oscillatory response	Yes	Weak	No
Robust interval	Not imposed	Design-dependent	μ from 0.2 to 0.8
<i>Note – compiled by the authors from simulated $x_1(t)$ over 0-10 s.</i>			

The gradient-velocity realization reaches the 2% band in 3.07 s and has zero undershoot. The classical benchmark requires 6.35 s and reaches -28.2% of the initial x_1 value. The LMI benchmark has the same 3.07 s settling time but retains a small 1.4% undershoot.

The IAE values are 1.415, 1.213, and 0.784 for the classical, LMI, and gradient-velocity realizations, respectively. For this specific benchmark set, the non-oscillatory model also produces the smallest accumulated absolute error.

These values demonstrate the intended transient property of the selected realization. They do not establish universal superiority over all LMI or classical designs, which can be tuned with additional pole-region and performance constraints.

Table 2 clarifies the scope of the proposed framework. Its distinctive output is not merely a negative Lyapunov derivative, but a separate statement about the shape of the transient response for the chosen communication-error coordinates.

Table 2. Comparative characteristics of classical and gradient-velocity stability approaches

Characteristic	Classical quadratic Lyapunov	LMI-based robust design	Gradient-velocity framework
Primary objective	Asymptotic stability	Stability and performance through convex constraints	Robust decay plus explicit aperiodicity screening
Uncertainty handling	Depends on the selected common function	Polytopic or norm-bounded sets	Affine interval checked at its endpoints
Transient information	Not inherent in $dV/dt < 0$	Available when pole or performance constraints are added	Sign preservation and monotonicity checked for the selected realization

End of table 2

Characteristic	Classical quadratic Lyapunov	LMI-based robust design	Gradient-velocity framework
Computation	Lyapunov equation or inequality	Semidefinite program	Endpoint inequalities and time-response verification
Main limitation	Limited direct transient information	Controller- and formulation-dependent complexity	Aperiodicity statement is realization-dependent
<i>Note – compiled by the authors from the analytical comparison in this study</i>			

Figure 3 shows the geometric difference directly. The classical trajectory changes sign several times. The LMI benchmark strongly suppresses this behavior but still crosses zero once, whereas the gradient-velocity trajectory remains positive and monotonic.

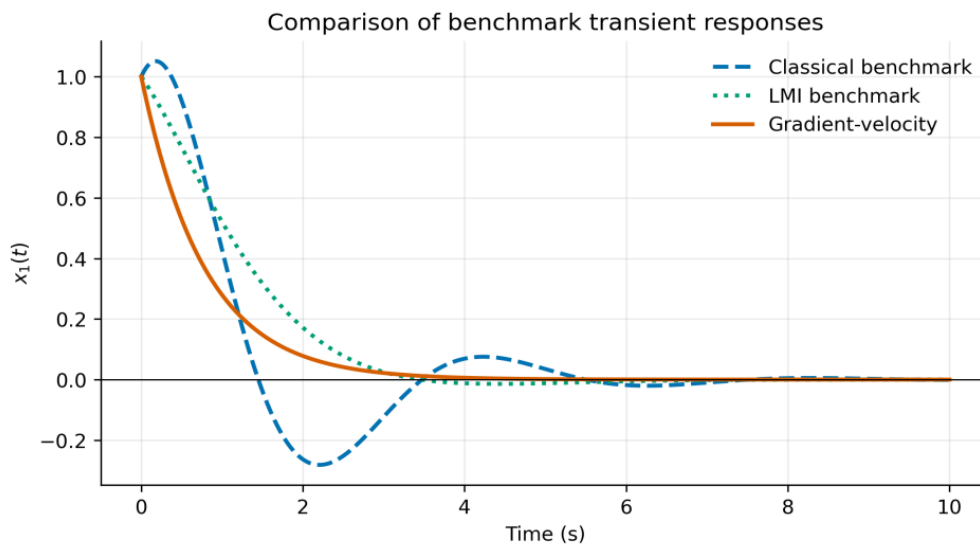


Figure 3. Comparison of benchmark transient responses

Note – compiled by the authors

The absence of a sign reversal is consistent with the real negative poles of the diagonal realization. This property is useful when repeated corrections or overshoot are undesirable during tracking-loop acquisition.

The comparison therefore supports a narrower conclusion: the selected gradient-velocity realization meets the imposed aperiodicity requirement while retaining a settling time comparable to the faster oscillatory benchmark.

ROBUSTNESS ANALYSIS

The uncertain parameter is swept across four interior and endpoint values. Because the matrix is diagonal, increasing μ in this normalized realization increases the effective decay rate; μ should therefore be interpreted as an effective loop-gain index rather than as a severity scale.

Figure 4 confirms robust sign-preserving convergence for $\mu=0.2, 0.4, 0.6$, and 0.8 . The slowest case occurs at $\mu=0.2$, where the x_1 decay rate is 1.17 s^{-1} and the 2% settling time is 3.34 s . At $\mu=0.8$ the rate is 1.38 s^{-1} and the settling time decreases to 2.83 s .

The numerical sweep agrees with the endpoint certificate: the symmetric matrix $A(\mu)^T + A(\mu)$ is bounded above by $-2.34I_2$. The result covers every μ in the interval, not only the four plotted values, because the symmetric matrix depends affinely on μ .

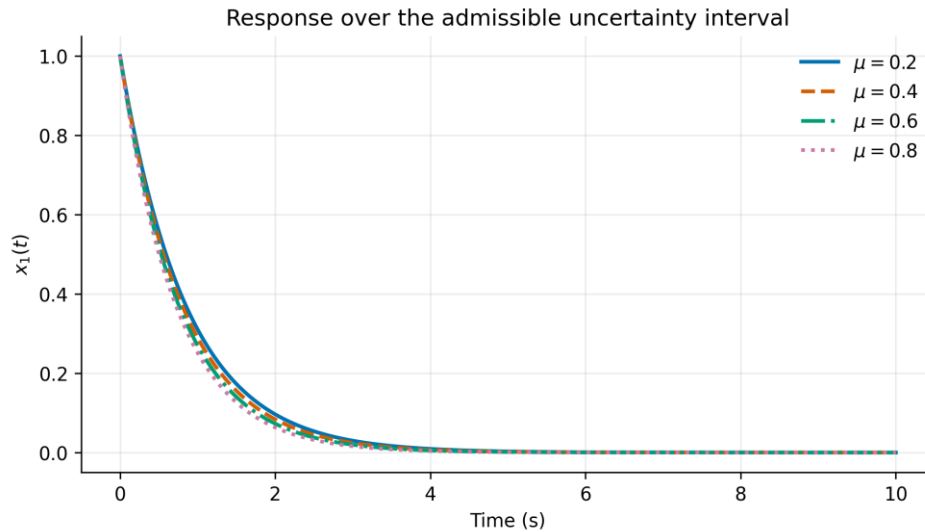


Figure 4. Response over the admissible uncertainty interval

Note – compiled by the authors

THIRD-ORDER VALIDATION

The third-order model adds carrier-phase error while preserving the same uncertainty structure. At $\mu=0.5$ the three decay rates are 1.125, 1.40, and 1.725 s^{-1} . Figure 5 shows that all three normalized errors remain positive and decrease without oscillation.

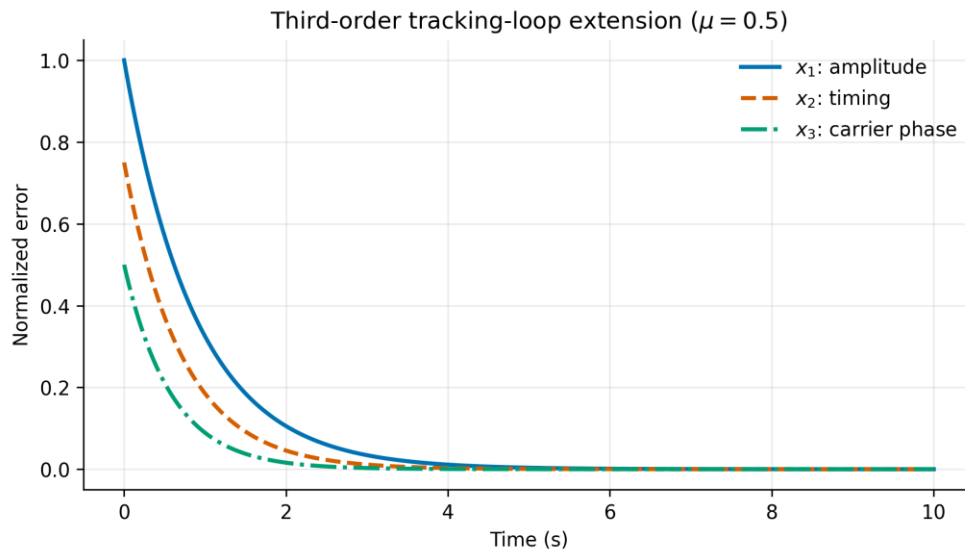


Figure 5. Third-order tracking-loop extension at $\mu=0.5$

Note – compiled by the authors.

The slowest state again determines the settling time. The analytical values in Table 3 follow from $x_1(t) = \exp[-(1.00 + 0.25\mu)t]$.

Table 3. Third-order model performance over the uncertainty interval

μ	Decay margin α	Settling time, s	IAE (0-10 s)	Undershoot, %
0.2	2.10	3.73	0.952	0.0
0.5	2.25	3.48	0.889	0.0
0.8	2.40	3.26	0.833	0.0

Note – compiled by the authors from the analytical solution for $x_1(t)$.

The third-order example has a positive common decay margin at every tested point and zero undershoot. It therefore supports the dimensional extension of the verification procedure, although it does not by itself establish scalability to strongly coupled or high-order modern models.

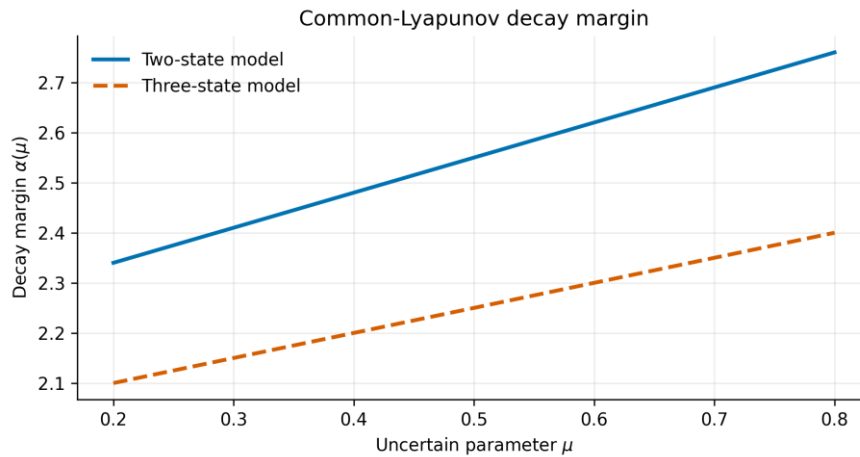


Figure 6. Common-Lyapunov decay margin for the two model orders

Note – compiled by the authors

Figure 6 shows that the endpoint-certified margin remains at least 2.34 for the two-state model and 2.10 for the third-order model for every μ in $[0.2, 0.8]$.

SENSITIVITY OF THE PERFORMANCE INDICATORS

The endpoint certificate establishes stability over the entire interval, but it does not show how the transient indicators change inside that interval. For the two-state diagonal realization, the slow component is $x_1(t) = \exp[-r(\mu)t]$, where $r(\mu) = 1.10 + 0.35\mu$. Its 2% settling time is therefore $t_s(\mu) = \frac{\ln(50)}{r(\mu)}$, while the finite-horizon integral absolute error is $IAE_1(\mu) = \frac{1 - \exp(-10r(\mu))}{r(\mu)}$. Both expressions are continuous and strictly decreasing throughout the interval $[0.2, 0.8]$. Consequently, no unexamined interior value of the parameter can produce a slower decay than the lower endpoint for this realization.

Table 4 reports a denser seven-point sweep. The decay margin increases by 17.9% between the interval endpoints, whereas the settling time decreases by 15.2% and IAE_1 decreases by 15.2%. These monotone changes are expected because μ strengthens the negative diagonal feedback in the adopted uncertainty direction. This trend must not be generalized to arbitrary uncertainty: if an uncertain coefficient weakens damping or changes the off-diagonal coupling, the worst case may move to another vertex or require a polytopic certificate with more than two vertices.

Table 4. Sensitivity of the two-state model to the uncertainty parameter

μ	Slow decay rate, s^{-1}	Decay margin α	Settling time, s	IAE ₁ (0-10 s)
0.2	1.170	2.34	3.34	0.855
0.3	1.205	2.41	3.25	0.830
0.4	1.240	2.48	3.15	0.806
0.5	1.275	2.55	3.07	0.784
0.6	1.310	2.62	2.99	0.763
0.7	1.345	2.69	2.91	0.743
0.8	1.380	2.76	2.83	0.725

Note – calculated by the authors from the analytical solution of the slow state; the 2% criterion is used for settling time

The analytical sweep also provides a reproducibility check for the numerical integration. At $\mu=0.5$, for example, the slow rate is 1.275 s^{-1} , which gives $t_s = 3.07\text{ s}$ and $IAE_1 = 0.784$. These values coincide with Table 1 to the shown precision. The comparison is useful because it separates integration or post-processing errors from genuine changes in the model. For a coupled realization, closed-form state metrics may be unavailable; the same check can then be made against the matrix exponential $\exp(A(\mu)t)$ at selected time instants.

SCOPE OF THE ENDPOINT ARGUMENT

The sufficiency of endpoint tests follows from convexity, not from the numerical diagonal form alone. Let $S(\mu) = A^T(\mu)P + PA(\mu)$, with P symmetric positive definite and $A(\mu)$ affine. For any μ between μ_{min} and μ_{max} there is a coefficient θ between 0 and 1 such that $S(\mu) = \theta S(\mu_{min}) + (1 - \theta)S(\mu_{max})$. If both endpoint matrices are negative definite with a common margin, their convex combination is also negative definite. This proves robust quadratic stability for every constant parameter value in the interval and avoids replacing a proof by a finite simulation grid.

The same reasoning extends to several independent affine parameters by checking all vertices of the resulting uncertainty polytope with one common P . The number of vertices grows exponentially with the number of independent parameters, so direct enumeration is most practical for low-dimensional uncertainty sets. Structured multipliers, parameter-dependent Lyapunov functions, or randomized validation may be considered when the vertex set becomes large. Such alternatives can reduce conservatism or computation, but they change the certificate and are outside the numerical claim made in this study.

Quadratic stability should also be distinguished from aperiodicity. A common P proves decay of an energy-like norm; it does not, by itself, forbid individual state components from changing sign. In the reported second- and third-order examples, zero undershoot follows additionally from the diagonal real-negative structure and positive initial conditions. For a coupled model, a real negative spectrum is not always sufficient to preserve component signs because eigenvector combinations can cross zero. Positivity conditions, invariant-cone analysis, or direct componentwise verification must then supplement the Lyapunov test.

REPRODUCIBLE APPLICATION TO A TRACKING-LOOP MODEL

Application to a specific satellite terminal should begin with an identified small-signal model rather than with a chosen matrix. Timing, amplitude, and carrier-phase error signals can be perturbed separately around a locked operating point, and the local state matrix can be estimated for representative Doppler rate, signal-to-noise ratio, oscillator offset, and loop gain settings. The normalization used in this paper should be recorded explicitly so that a unit initial state can be mapped back to physical timing, amplitude, or phase error.

Next, the observed variation should be enclosed by an affine interval or polytope $A(q) = A_0 + \sum q_i \Delta A_i$, where the sum is taken over all independent uncertain parameters. The enclosure must be validated against the identified matrices; otherwise the subsequent certificate applies to a mathematical family that may not cover the terminal. A common P and the largest feasible decay margin can then be obtained from the vertex LMIs. Reporting P , every vertex matrix, solver tolerances, and the minimum eigenvalue of each residual makes the stability calculation independently checkable.

The transient test should include every vertex and a sufficiently dense interior sweep, but the roles of these operations are different. Vertex LMIs provide the robust-stability proof for the affine family; time simulations reveal performance and possible sign changes. Settling time, IAE, peak error, and the number of zero crossings should be evaluated under the same initial conditions and horizon for all compared designs. Delay, saturation, noise, and time-varying parameters should then be introduced one at a time to show which assumption first invalidates the nominal certificate.

Finally, an experimental validation should compare predicted and measured error envelopes after controlled acquisition or reacquisition events. The proposed framework would be supported if the identified matrices remain inside the certified set, the measured states converge without the prohibited zero crossings, and observed settling times lie within the predicted bounds after accounting for sampling and estimation error. This staged procedure connects the present proof-of-concept model to link-level evidence without claiming that the normalized numerical example already represents a complete satellite modem.

DISCUSSION AND SCIENTIFIC CONTRIBUTION

The numerical results support the central claim at the level at which it is formulated: for the stated affine tracking-loop models, one common quadratic certificate proves robust decay, and the diagonal negative realization proves componentwise aperiodicity. This separation is methodologically important because robust asymptotic stability and a non-oscillatory transient are related but not equivalent properties.

The benchmark comparison also reveals the relevant trade-off. Complex poles can provide fast decay while introducing zero crossings; real negative poles can remove those crossings but do not automatically optimize every performance criterion. In the present numerical set the gradient-velocity realization achieves both the lowest IAE and zero undershoot, but this outcome is specific to the prescribed matrices. LMI formulations can incorporate decay-rate or pole-region constraints when a full controller-synthesis problem is posed (Boyd et al., 1994; Brusnicki et al., 2026).

The communication interpretation is consistent with work showing that synchronization and channel uncertainty influence LEO link performance (Kim & Park, 2023; Liu et al., 2022; Alsenwi et al., 2024). Nevertheless, amplitude, timing, and carrier errors are represented here as normalized states. The study does not include a radio-frequency front end, coding, fading statistics, or an identified controller from a specific satellite terminal. Practical claims are therefore limited to preliminary tracking-loop analysis.

Several limitations define the next stage of the work. The uncertainty is scalar and constant within a run; delays, stochastic disturbances, saturation, measurement noise, and coupling between loops are omitted. Future work should identify $A(\mu)$ from measured or high-fidelity modem data, extend the certificate to polytopic and time-varying uncertainty, include delay-dependent conditions, and compare controller designs on one common plant. These steps are required before conclusions about link-level reliability can be made.

CONCLUSION

This study presented a gradient-velocity framework for the aperiodic robust-stability analysis of normalized satellite-communication tracking loops with affine parametric

uncertainty. The framework combines a common quadratic Lyapunov certificate with a separate sign-preservation argument for the state trajectories.

For the corrected second-order model, $P = I_2$ provides a common decay margin of at least 2.34 for every μ in $[0.2, 0.8]$. The nominal trajectory settles in 3.07 s with IAE = 0.784 and no undershoot. A third-order extension also remains robustly stable and monotonic, with a common margin of at least 2.10.

The comparison with two stable oscillatory benchmarks shows the practical role of the aperiodicity requirement. The result is not a universal ranking of Lyapunov, LMI, and gradient-velocity designs; it is a reproducible demonstration that stability and transient shape should be certified separately.

The present models are normalized and illustrative. Experimental validation, identified modem parameters, coupled dynamics, time delays, and stochastic channel effects are outside the current scope and constitute the main directions for further research.

Accordingly, the proposed framework is best viewed as a preliminary-design and verification tool for communication tracking loops rather than as a complete satellite-link model.

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